Electromagnetic fields in a three-dimensional cavity and in a waveguide with oscillating walls

1998 J. Phys. A: Math. Gen. 31 L457

(http://iopscience.iop.org/0305-4470/31/24/003)

J. Phys. A: Math. Gen. **31** (1998) L457–L462. Printed in the UK PII: S0305-4470(98)92253-3

LETTER TO THE EDITOR

Electromagnetic fields in a three-dimensional cavity and in a waveguide with oscillating walls

Jeong-Young Ji†‖, Kwang-Sup Soh†¶, Rong-Gen Cai‡+ and Sang Pyo Kim§*

† Department of Physics Education, Seoul National University, Seoul 151-742, Korea
‡ Center for Theoretical Physics, Seoul National University, Seoul, 151-742, Korea
§ Department of Physics, Kunsan National University, Kunsan 573-701, Korea

Received 5 March 1998

Abstract. We consider classical and quantum electromagnetic fields in a three-dimensional cavity and in a waveguide with oscillating boundaries of the frequency Ω. The photons created by the parametric resonance are distributed in the wavenumber space around $\Omega/2$ along the axis of the oscillation. When classical waves propagate along the waveguide in the one direction, we observe the amplification of the original waves and another wave generation in the opposite direction by the oscillation of side-walls. In the case of two opposite walls oscillating with the same frequency but with a phase difference, the interferences are shown to occur because of the phase difference in the photon numbers and in the intensity of the generated waves.

The Casimir effect [1] is a macroscopic manifestation of the change in the zero-point electromagnetic energy due to the walls. The time-varying boundary conditions induce the change of the vacuum states for the quantum electromagnetic fields and the difference between the initial and final vacuum states results in the photon production. This dynamical Casimir effect provides the possibility to experimentally observe the vacuum change of quantum fields. This phenomenon has been extensively studied when one of the walls oscillates [2–5]. For an almost sinusoidal movement of the mirror, using the formalism invented by Moore [5] and developed by Fulling and Davies [6], the quantum energy density has been calculated [7, 8]. For a harmonic oscillation of the mirror Méplan and Gignoux have shown that a set of frequencies of the oscillating walls leads to an exponential growth of the energy of a wave [9] and the exponential growth of the number of generated photons can be easily understood from Floquet's theorem [10]. The scattering approach is used to analyse the motion-induced radiation from a vibrating cavity with partly transmitting mirror(s) [11, 12]. In a classical periodically driven string, the existence of instability (in the sense of unlimited growth of energy) is proven and demonstrated in [13]. The phenomenon of photon production has also been investigated in three-dimensional (3D) cavity by using different methods [14–16]. When the oscillation amplitude of walls is small, in the previous paper [17] we have developed a perturbation method to calculate the time evolution of the electromagnetic field in the instantaneous basis [18–21].

‖ E-mail address: jyji@phyb.snu.ac.kr
¶ E-mail address: kssoh@phya.snu.ac.kr
+ E-mail address: cairg@ctp.snu.ac.kr
* E-mail address: sangkim@knusun1.kunsan.ac.kr

0305-4470/98/240457+06$19.50

In this letter, using the perturbation method, we wish to consider the photon production in a 3D cavity and to discuss the propagation of classical electromagnetic fields in a waveguide with oscillating walls. The photons created by the parametric resonance are distributed in the wavenumber space around the half of the oscillation frequency along the axis of the oscillating motion. We find that if one transmits the classical waves into the waveguide with oscillating walls, the waves are amplified and there are generated waves propagating in the opposite direction, which corresponds to the photon production in the quantum theory. When two walls oscillate a kind of interference phenomena take place in the photon numbers and in the intensity of the generated waves.

In the three dimensions, the electromagnetic field has two directions of physical polarization. For simplicity, assuming that the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ is polarized in the z-direction, we can write down [5]

$$\begin{aligned} \boldsymbol{A} &= A(x, y, t)\hat{\boldsymbol{z}} \\ \boldsymbol{E} &= E\hat{\boldsymbol{z}} = -\frac{\partial A}{\partial t}\hat{\boldsymbol{z}} \\ \boldsymbol{B} &= \frac{\partial A}{\partial y}\hat{\boldsymbol{x}} - \frac{\partial A}{\partial x}\hat{\boldsymbol{y}}. \end{aligned} \tag{1}$$

Consider a rectangular cavity with sides $q_x(t)$, L_y and L_z, where one of the walls oscillates for a time interval $0 < t < T$ with a small amplitude ($\epsilon \ll 1$) according to

$$q_x(t) = L_x(1 + \epsilon \sin \Omega t). \tag{2}$$

In this cavity the field operator can be expanded

$$A = \sum_{\boldsymbol{n}} [b_{\boldsymbol{n}} \psi_{\boldsymbol{n}} + b_{\boldsymbol{n}}^{\dagger} \psi_{\boldsymbol{n}}^{*}] \tag{3}$$

using the following instantaneous basis

$$\psi_{\boldsymbol{n}}(x, y|q_x(t)) = \sum_{\boldsymbol{k}} Q_{\boldsymbol{nk}} \varphi_{\boldsymbol{k}}(x, y, t) \tag{4}$$

where

$$\varphi_{\boldsymbol{k}}(x, y|q_x(t)) = \frac{2}{\sqrt{q_x L_y L_z}} \sin \frac{\pi k_x x}{q_x} \sin \frac{\pi k_y y}{L_y} \tag{5}$$

with $k_x, k_y = 1, 2, 3, \ldots$. From Maxwell's equations or the wave equation for $\psi_{\boldsymbol{n}}$ we have

$$\begin{aligned} \ddot{Q}_{\boldsymbol{nk}} &= -\omega_{\boldsymbol{k}}^2 Q_{\boldsymbol{nk}} + 2\epsilon(\pi k_x/L_x)^2 \sin \Omega t\, Q_{\boldsymbol{nk}} \\ &\quad + 2\epsilon\Omega \cos \Omega t \sum_{\boldsymbol{j}} g_{\boldsymbol{kj}} \dot{Q}_{\boldsymbol{nj}} - \epsilon\Omega^2 \sin \Omega t \sum_{\boldsymbol{j}} g_{\boldsymbol{kj}} Q_{\boldsymbol{nj}} + \mathrm{O}(\epsilon^2) \end{aligned} \tag{6}$$

with

$$g_{\boldsymbol{jk}} = (-1)^{j_x + k_x} \frac{2 j_x k_x}{k_x^2 - j_x^2} \delta_{j_y k_y} \qquad (j_x \neq k_x) \tag{7}$$

and $g_{\boldsymbol{jk}} = 0$ for $j_x = k_x$. Using the perturbation method developed in [17], the solution can be written as

$$Q_{\boldsymbol{nk}} = Q_{\boldsymbol{nk}}^{(0)} + \epsilon Q_{\boldsymbol{nk}}^{(1)} + \cdots \tag{8}$$

where

$$Q_{\boldsymbol{nk}}^{(0)}(t) = \frac{\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}} t}}{\sqrt{2\omega_{\boldsymbol{k}}}} \delta_{\boldsymbol{nk}} \tag{9}$$

and

$$Q_{nk}^{(1)}(t) = \sum_{\sigma,s=\pm} w_{k\sigma,n-}^{s} \frac{e^{\sigma i\omega_k t}}{\sqrt{2\omega_k}} \int_0^t dt'\, e^{-i(\sigma\omega_k - s\Omega + \omega_n)t'} \tag{10}$$

with

$$w_{k\sigma,n\sigma'}^{s} = \sigma \left[\Omega g_{kn} \sqrt{\frac{\omega_n}{\omega_k}} \left(\frac{s\Omega}{4\omega_n} + \frac{\sigma'}{2} \right) - s \frac{(k_x\pi/L_x)^2}{2\omega_k} \delta_{kn} \right]. \tag{11}$$

Here we note that the zeroth-order solution describes the field operator at the static situation:

$$A = \sum_n [b_n \phi_n + b_n^\dagger \phi_n^*] \tag{12}$$

where $\phi_n = \frac{1}{\sqrt{2\omega_n}} e^{-i\omega_n t} \varphi(x, y|L_x)$. After some interval T of the oscillation of the wall, the Heisenberg field operator can be written as

$$A = \sum_n [a_n \phi_n + a_n^\dagger \phi_n^*] \tag{13}$$

where

$$a_k = \sum_n [b_n \alpha_{nk} + b_n^\dagger \beta_{nk}^*] \tag{14}$$

with

$$\sum_n (|\alpha_{nk}|^2 - |\beta_{nk}|^2) = 1. \tag{15}$$

The Bogoliubov coefficient β_{nk} can be read from the solution $Q_{nk}^{(1)}$ to the leading order in ϵ by retaining dominant terms only ($\omega T \gg 1$):

$$\beta_{nk} = \epsilon T w_{k+,n-}^{+} \delta_{\omega_n, \Omega-\omega_k}. \tag{16}$$

Hereafter we introduce the bar notation for the wavenumber vector: $\bar{n}$ denotes the wavenumber vector corresponding to n or in the components, $(\bar{n}_x, \bar{n}_y) = (n_x\pi/L_x, n_y\pi/L_y)$. Noting that the resonance conditions

$$\omega_n = \Omega - \omega_k \qquad \text{and} \qquad \bar{n}_y = \bar{k}_y \tag{17}$$

can be explicitly written as†

$$\bar{n}_x^2 = \bar{k}_x^2 - 2\Omega\omega_k + \Omega^2 \tag{18}$$

we have the following number distribution in the created photons:

$$N_k = \sum_n |\beta_{nk}|^2 = \left(\frac{\epsilon T}{2} \right)^2 \frac{\bar{k}_x^2 [\bar{k}_x^2 - 2\Omega\omega_k + \Omega^2]}{\omega_k(\Omega - \omega_k)}. \tag{19}$$

This distribution of created photons is anisotropic in the wavenumber space (see figure 1) and the number of created photons is maximal at the nearest neighbour of

$$\bar{k}_x = \frac{\Omega}{2} \qquad \text{and} \qquad \bar{k}_y = 0 \tag{20}$$

† There may not exist an integer n_x satisfying (18) because the discreteness of the frequency and in this case no photons are created by the parametric resonance. However, for the case $L_x \ll L_y$, the frequency can be regarded as a continuum and in this case the resonance condition will be fulfilled by an integer n_x. Further, the condition of parametric resonance admits some discrepancy as seen from the solutions of the Mathieu equation.

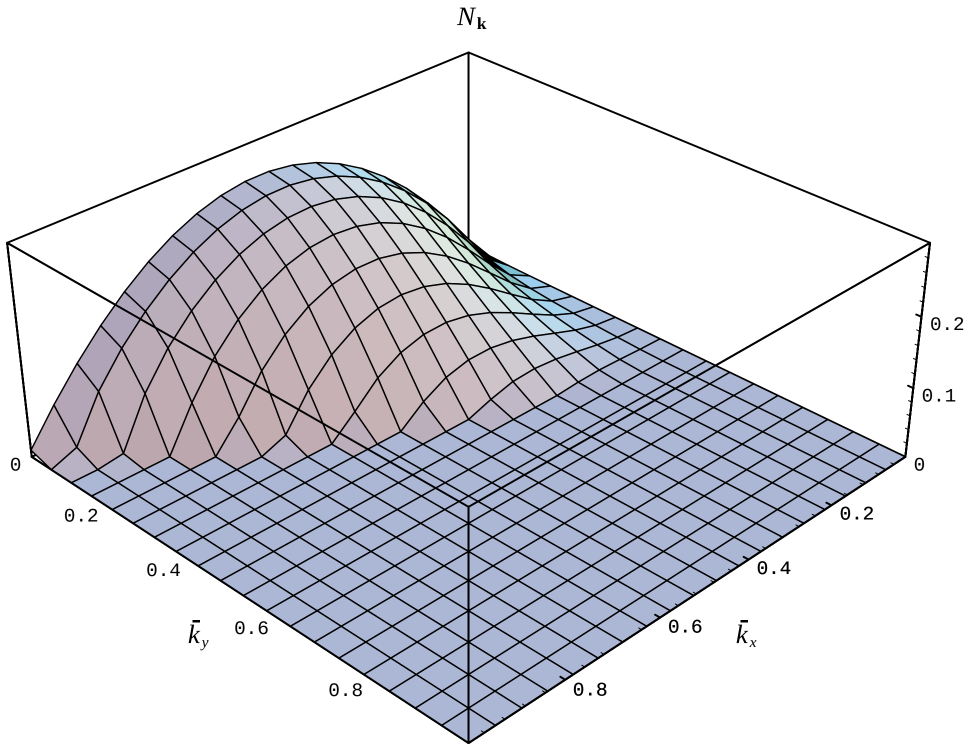

Figure 1. The distribution of created photons in the rectangular cavity when the side at $x = q_x(t)$ oscillates sinusoidally with a small amplitude. $N_{\boldsymbol{k}}$ is denoted in units of $(\epsilon\Omega T/2)^2$, and $\bar{k}_x$ and $\bar{k}_y$ are denoted in units of Ω.

which comes from the conditions $\partial N_{\boldsymbol{k}}/\partial\bar{k}_x = 0$ and $\partial N_{\boldsymbol{k}}/\partial\bar{k}_y = 0$. In fact, the second condition in (20) means that $k_y = 1$ because $k_y = 0$ means the vanishing field. Note that the maximum number of photons are created for $\omega_{\max} = \Omega/2$ which is the characteristic condition of the parametric resonance near the axis of oscillation (with the minimum $\bar{k}_y$).

Now we extend the results to the case when the left and right walls oscillate with frequencies Ω_L and Ω_R respectively:

$$N_{\boldsymbol{k}} = N_{\boldsymbol{k}}^L + N_{\boldsymbol{k}}^R - (-1)^{k_x+n_x} 2\sqrt{N_{\boldsymbol{k}}^L}\sqrt{N_{\boldsymbol{k}}^R}\cos\phi\,\delta_{\Omega_L,\Omega_R} \tag{21}$$

where $N_{\boldsymbol{k}}^L$ and $N_{\boldsymbol{k}}^R$ are obtained by replacing Ω in (19) with Ω_L and Ω_R, respectively. Here ϕ is the initial phase difference between two oscillations of the walls and n_x is a positive integer satisfying (18) (see the earlier footnote). When $\Omega_L \neq \Omega_R$, the number of generated photons by the parametric resonance is the sum of the photon numbers generated when the left and right walls oscillate separately. When $\Omega = \Omega_L = \Omega_R$, there is an additional interference term depending on the mode number of x-component and the phase difference ϕ. This is just the interference phenomenon found in the one-dimensional (1D) case [22]. It is worth noting that when $\phi = 0$ or $\phi = \pi$ whether the interference is constructive or destructive is determined by x-component mode numbers only. Unlike the 1D case, $n_x + k_x$ does not represent the ratio of the oscillation frequency Ω to the fundamental mode frequency $\omega_{(1,1)} = \sqrt{(\pi/L_x)^2 + (\pi/L_y)^2}$. However, for $L_y \gg L_x$ and $\bar{k}_y \approx 0$, $n_x + k_x$ is an integer close to $\gamma = \Omega/\omega_{(1,1)}$. In this case we have a constructive (destructive) interference when

$\phi = \pi$ (for $\gamma = 2, 4, \ldots \gamma = 1, 3, \ldots$) and when $\phi = 0$ (for $\gamma = 1, 3, \ldots \gamma = 2, 4, \ldots$). For a general 3D mode, it is possible to have a constructive (destructive) interference when $\phi = \pi$ for an odd (even) γ since the mode frequency depends not only on the x-mode number but also the y-mode number.

We now turn to the classical electromagnetic waves propagating in the positive y-direction in the rectangular cavity with an oscillating wall. When the wall of a cavity is static the right-going wave is described by

$$A(x, y, t) = \sum_{\boldsymbol{n}} f_{\boldsymbol{n}} \mathcal{N}_{\boldsymbol{n}} \cos(\bar{n}_y y - \omega_{\boldsymbol{n}} t) \sin \bar{n}_x x \tag{22}$$

with the normalization constant $\mathcal{N}_{\boldsymbol{n}} = \sqrt{2/\omega_{\boldsymbol{n}} \pi L_x L_z}$. Here the sum over $\boldsymbol{n}$ denotes the sum over $\bar{n}_x = n_x \pi / L_x$ ($n_x = 1, 2, 3, \ldots$) and the integration over $\bar{n}_y$, and $f_{\boldsymbol{n}}$ is a real distribution function of the wavenumber vector $\bar{n} = \bar{n}_x \hat{\boldsymbol{x}} + \bar{n}_y \hat{\boldsymbol{y}}$. There is no boundary along the y-direction and $\bar{n}_y$ can be regarded as a continuum limit of $n_y \pi / L_y$ ($L_y \to \infty$). Introducing the propagating instantaneous basis

$$\varphi_{\boldsymbol{k}}(x, y | q_x(t)) = \frac{1}{\sqrt{\pi q_x(t) L_z}} \sin \frac{\pi k_x x}{q_x(t)} \mathrm{e}^{\mathrm{i} \bar{k}_y y} \tag{23}$$

the classical vector potential at any time can be expanded as

$$A(x, y, t) = \sum_{\boldsymbol{n}} \left[f_{\boldsymbol{n}} \sum_{\boldsymbol{k}} (Q_{\boldsymbol{nk}} \varphi_{\boldsymbol{k}} + \text{h.c.}) \right]. \tag{24}$$

With the initial condition (9) and the static-wall solution with $q_x(t) = L_x$ in (23), the field (24) becomes (22). The time evolution of $Q_{\boldsymbol{nk}}(t)$ is given by solving the same equation (6) and we have the same solution. Then for the oscillating wall we have the following wave:

$$A(x, y, t) = \sum_{\boldsymbol{n},\boldsymbol{k}} f_{\boldsymbol{n}} \alpha_{\boldsymbol{nk}} \mathcal{N}_{\boldsymbol{k}} \cos(\bar{k}_y y - \omega_{\boldsymbol{k}} t) \sin \bar{k}_x x + \sum_{\boldsymbol{n},\boldsymbol{k}} f_{\boldsymbol{n}} \beta_{\boldsymbol{nk}} \mathcal{N}_{\boldsymbol{k}} \cos(\bar{k}_y y + \omega_{\boldsymbol{k}} t) \sin \bar{k}_x x \tag{25}$$

where $\beta_{\boldsymbol{nk}}$ is given by (16) to the leading order in ϵ. Thus we have the left-going wave in the y-direction induced by the x-directional oscillation of the wall, with the amplitude of the $\boldsymbol{k}$th mode being proportional to $\sum_{\boldsymbol{n}} f_{\boldsymbol{n}} \beta_{\boldsymbol{nk}}$ with $|\sum_{\boldsymbol{n}} f(\bar{\boldsymbol{n}}) \beta_{\boldsymbol{nk}}|^2 = f(\sqrt{\bar{k}_x^2 - 2\Omega\omega_{\boldsymbol{k}} + \Omega^2}, \bar{k}_y) N_{\boldsymbol{k}}$, where $f(\bar{\boldsymbol{n}})$ denotes $f_{\boldsymbol{n}}$ and $N_{\boldsymbol{k}}$ is given by equation (19). When the two side-walls oscillate, we find the interference phenomena again as in the case of the quantum field.

Before proceeding to analyse our results, let us survey some related works for the 3D cavity with oscillating walls. It has been suggested that a fantastic amount of photons can be generated in a 3D cavity with one plate being performed periodic instantaneous jumps between two stationary positions [3]. However, this result was obtained by neglecting the terms coupled to other frequency modes in equation (6), as pointed out in [21], and the cases not satisfying equation (35) in [3] which corresponds to the condition of parametric resonance [23] were neglected. In fact, such a large number can be obtained from the ultraviolet divergence for the instantaneous jump of the frequency. In [19, 21], the 3D problem was reduced to a decoupled single parametric oscillator. Using the ansatz $Q_{\boldsymbol{k}} = \xi_{\boldsymbol{k}}(\epsilon t) \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{k}} t} + \eta_{\boldsymbol{k}}(\epsilon t) \mathrm{e}^{\mathrm{i}\omega_{\boldsymbol{k}} t}$ together with the assumption that $\xi_{\boldsymbol{k}}$ and $\eta_{\boldsymbol{k}}$ are slowly varying functions of time, and averaging over fast oscillations, the coupling terms (second line in equation (6)) have been neglected again in the effective theory. In our case, the coupling terms are also considered and they affect the calculation of the Bogoliubov coefficient $\beta_{\boldsymbol{nk}}$.

One may wonder if the resonance condition (18) cannot be satisfied because of the discreteness of the frequency in the cavity. As stated in the earlier footnote a small deviation from the resonance condition is admitted and it can be satisfied in the continuum limit $(L_x \ll L_y)$. Thus, the propagating wave in the waveguide $(L_y \rightarrow \infty)$ provides a good experimental situation to observe the generation of the wave or the photon production. The intensity of the generated wave is the order of the intensity of the incident wave multiplied by the number of produced photons in the quantum theory. After a very short time, we may observe an amount of created left-going wave if we prepare the incident right-going wave satisfying the resonance condition (17). For an experimental situation to observe N photons/s ($N = 10$ in [12], $N = 600$ in [21]), it takes only $1/N$ s to obtain the same intensity of the generated wave as the incident wave, then the incident right-going wave will also be amplified by the order of $1 + N$ according to (15). This phenomenon may be regarded as the classical counterpart of the photon production in the quantum theory.

This work was supported by the Center for Theoretical Physics (SNU), and the Basic Science Research Institute Program, Ministry of Education project no BSRI-97-2418. JYJ would like to thank Dr J H Cho for very helpful discussions.

References

[1] Casimir H B G 1948 *Proc. K. Ned. Akad. Wet.* **51** 793
[2] Dodonov V V, Klimov A B and Man'ko V I 1990 *Phys. Lett.* A **149** 225
[3] Sassaroli E, Srivastava Y N and Widom A 1994 *Phys. Rev.* A **50** 1027
[4] Klimov A B and Altuzar V 1997 *Phys. Lett.* A **226** 41
[5] Moore G T 1970 *J. Math. Phys.* **11** 2679
[6] Fulling S and Davies P 1976 *Proc. R. Soc.* A **348** 393
[7] Law C K 1994 *Phys. Rev. Lett.* **73** 1931
[8] Cole C K and Schieve W C 1995 *Phys. Rev.* A **52** 4405
[9] Méplan O and Gignoux C 1996 *Phys. Rev. Lett.* **76** 408
[10] Ji J Y and Soh K S *J. Korean Phys. Soc.* to be published
[11] Jaekel M T and Reynaud S 1992 *Quantum Opt.* **4** 39
Jaekel M T and Reynaud S 1992 *J. Physique* I **2** 149
[12] Lambrecht A, Jaekel M-T and Reynaud S 1996 *Phys. Rev. Lett.* **77** 615
[13] Dittrich J, Duclos P and Seba P 1993 *Phys. Rev.* E **49** 3535
Dittrich J, Duclos P and Seba P 1994 *J. Operator Theor.* **70** 259
[14] Barton G and North C A 1996 *Ann. Phys., NY* **252** 72
[15] Mundarain D F and Maia Neto P A 1998 *Phys. Rev.* A **57** 1379
[16] Schutzhold R, Plunien Q and Soff G 1998 *Phys. Rev.* A **57** 2311
[17] Ji J Y, Jung H H, Park J W and Soh K S 1997 *Phys. Rev.* A **56** 4440
[18] Law C K 1995 *Phys. Rev.* A **51** 2537
[19] Dodonov V V 1995 *Phys. Lett.* A **207** 126
[20] Dodonov V V 1996 *Phys. Lett.* A **213** 219
[21] Dodonov V V and Klimov A B 1996 *Phys. Rev.* A **53** 2664
[22] Ji J Y, Jung H H and Soh K S 1997 *Phys. Rev.* A to be published
[23] Ji J Y and Kim J K 1995 *Phys. Lett.* A **208** 25